Limits and Continuity of Functions

Recall that the (Euclidean) distance between two points $\mathbf{x}$ and $\overline{\mathbf{x}}$ in $\mathbb{R}^n$ is given by the (Euclidean) norm, $\|\mathbf{x} - \overline{\mathbf{x}}\|$. Everything in this lecture will be based on this norm and the notion of distance it represents. Later in the course, when we begin to use alternative norms, most of the definitions and results for the Euclidean norm will carry over to the other norms. But we don't have to worry about that now.

Let ϵ be a positive real number. In $\mathbb{R}^2$ it's intuitive that a circle of radius ϵ with center at $\overline{\mathbf{x}}$ is the set of all the points whose distance from $\overline{\mathbf{x}}$ is ϵ. In $\mathbb{R}^3$ the set of points ϵ units away from a given point $\overline{x}$ is a sphere of radius ϵ centered at $\overline{x}$. We're going to be interested in the points *inside* a circle or sphere, generalized to $\mathbb{R}^n$, as in the following definition.

Definition: The (open) **ball of radius** ϵ about a point $\overline{\mathbf{x}} \in \mathbb{R}^n$ is the set $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \overline{\mathbf{x}}\| < \epsilon\}$, also called the ϵ-**ball about** $\overline{\mathbf{x}}$, and denoted $B(\overline{x}, \epsilon)$ or $B_\epsilon(\overline{x})$.

Note that in $\mathbb{R}^2$ the ϵ-ball $B(\overline{x}, \epsilon)$ is a disc with center at $\overline{x}$, and in $\mathbb{R}$ the ϵ-ball is an interval centered at $\overline{x}$.

Definition: A **neighborhood** of a point $\overline{x} \in \mathbb{R}^n$ is a set $\mathfrak{N}$ that contains an ϵ-ball about $\overline{x}$. (In particular, an ϵ-ball about $\overline{x}$ is itself a neighborhood of $\overline{x}$.)

Definition: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^n$, let $\overline{\mathbf{x}} \in \mathbb{R}^n$, and let $a \in \mathbb{R}$. If

$$\forall \epsilon > 0 : \exists \delta > 0 : \|\mathbf{x} - \overline{x}\| < \delta \Rightarrow |f(x) - a| < \epsilon, \tag{1}$$

we say that a is the limit of f at $\overline{x}$, which we write as $\lim_{x \to \overline{x}} f(x) = a$ or $\lim\limits_{x \to \overline{x}} f(x) = a$. If there is no a in $\mathbb{R}^n$ that satisfies (1), the limit of f at $\overline{x}$ does not exist.

Remark: If $\lim_{x \to \overline{x}} f(x)$ exists, it is unique.

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}$ is **continuous** at a point $\overline{x} \in \mathbb{R}^n$ if $\lim_{x \to \overline{x}} f(x) = f(\overline{x})$.

Remark: $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\overline{x}$ if and only if

$$\forall \epsilon > 0 : \exists \delta > 0 : \|\mathbf{x} - \overline{x}\| < \delta \Rightarrow |f(x) - f(\overline{x})| < \epsilon$$

— *i.e.*, if and only if

$$\forall \epsilon > 0 : \exists \delta > 0 : \mathbf{x} \in B(\overline{x}, \delta) \Rightarrow f(\overline{x}) \in B(f(\overline{x}, \epsilon)).$$

Example 1: Let $\lambda \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \lambda x$, and let $\overline{x} \in \mathbb{R}$. To verify that f is continuous at $\overline{x}$:

(i) If $\lambda = 0$ we can simply let δ be any positive number, since $|f(x) - f(\overline{x})| = 0 < \epsilon$ for any ϵ.

(ii) If $\lambda \neq 0$, then for any ϵ let $\delta = \frac{1}{|\lambda|}\epsilon$; then

$$|x - \overline{x}| < \delta \Leftrightarrow |x - \overline{x}| < \frac{1}{|\lambda|}\epsilon \Leftrightarrow |\lambda||x - \overline{x}| < \epsilon \Leftrightarrow |\lambda x - \lambda \overline{x}| < \epsilon \Leftrightarrow |f(x) - f(\overline{x})| < \epsilon.$$

Example 2: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \leqq 1 \\ x - 1, & \text{if } x > 1. \end{cases}$$

Then f is discontinuous at $\overline{x} = 1$: we have $f(\overline{x}) = 1$, but let $\epsilon = \frac{1}{2}$, for example. Then for any $\delta > 0$ there are values of x for which $1 < x < 1 + \delta$ and $0 < f(x) < \frac{1}{2} = 1 - \epsilon = f(\overline{x}) - \epsilon$. See Figure 1. Therefore $|x - \overline{x}| < \delta$ and $|f(x) - f(\overline{x}|) > \epsilon$.

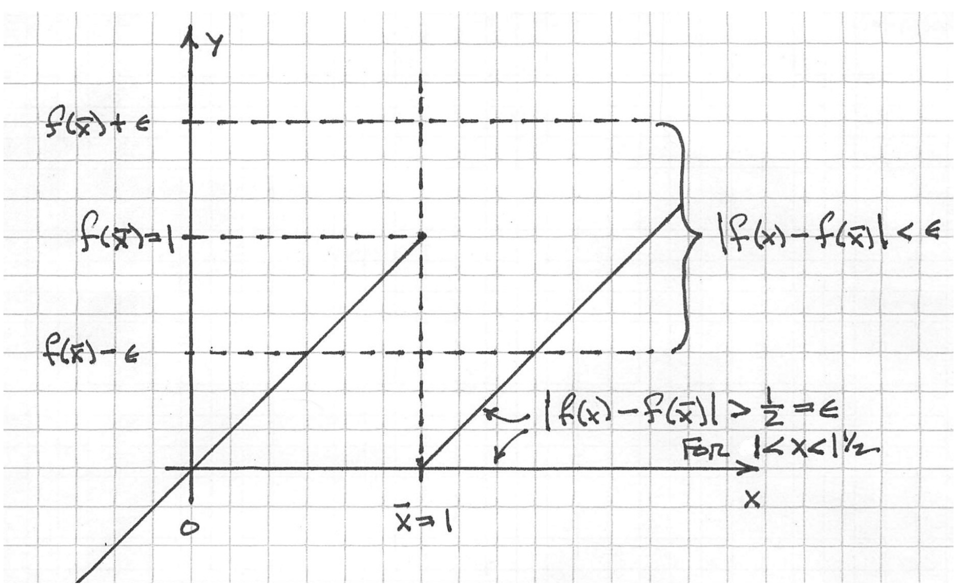

Figure 1