Example: The Pareto Maximization Problem

Let's assume we have a 2×2 exchange-only situation. We'll label the consumers as A and B and their consumption bundles as (x_A, y_A) and (x_B, y_B). Let's assume that each one's preference preordering is represented by the same utility function, $u(x, y) = x^2 y$, and assume that their total endowment of the two goods is the bundle $(\mathring{x}, \mathring{y}) = (25, 10)$.

(a) Let's check whether the allocation $((\hat{x}_A, \hat{y}_A), (\hat{x}_B, \hat{y}_B)) = ((15, 6), (10, 4))$ is a solution of the constrained maximization problem (P-max):

$$\begin{aligned} \max_{((x_A,y_A),(x_B,y_B))\in\mathbb{R}^4_+} \quad & u_A(x_A, y_A) = x_A^2 y_A \\ \text{subject to} \quad x_A + x_B &\leqq 25, \\ y_A + y_B &\leqq 10, \\ u_B(x_B, y_B) &\geqq u_B(\hat{x}_B, \hat{y}_B) = (10)^2 4 = 400. \end{aligned} \tag{P-Max}$$

We need to determine whether the numbers $\hat{x}_A = 15, \hat{y}_A = 6, \hat{x}_B = 10$, and $\hat{y}_B = 4$ satisfy the following equations for some numbers $\sigma_x, \sigma_y, \lambda \geqq 0$:

$$\begin{aligned} x_A &: & 2x_A y_A &= \sigma_x \\ y_A &: & x_A^2 &= \sigma_y \\ x_B &: & 0 &= \sigma_x - \lambda 2 x_B y_B \\ y_B &: & 0 &= \sigma_y - \lambda x_B^2 \\ \sigma_x &: & x_A + x_B &= 25 \\ \sigma_y &: & y_A + y_B &= 10 \\ \lambda &: & x_B^2 y_B &= 400. \end{aligned} \tag{FOC}$$

The last three equations are clearly satisfied at $((\hat{x}_A, \hat{y}_A), (\hat{x}_B, \hat{y}_B)) = ((15, 6), (10, 4))$. The first four equations are satisfied if $\sigma_x = 180, \sigma_y = 225$, and $\lambda = 9/4$. So $((\hat{x}_A, \hat{y}_A), (\hat{x}_B, \hat{y}_B))$ *is* a solution of (P-Max). (Notice that these were *equations*, not inequalities. The first four are equations because $\hat{x}_A, \hat{y}_A, \hat{x}_B$, and $\hat{y}_B$ are all positive — *i.e.*, the allocation $((\hat{x}_A, \hat{y}_A), (\hat{x}_B, \hat{y}_B))$ is *interior*. The last three have to be equations if σ_x, σ_y, and λ all turn out to be positive, as in fact they did.)

(b) Let's remind ourselves how we obtained the nice simple form of the (FOC) equations above. Let's write the three constraints in (P-Max) as follows:

$$\begin{aligned} G^1(x_A, y_A, x_B, y_B) &\leqq 25, \quad \text{where } G^1(x_A, y_A, x_B, y_B) := x_A + x_B \\ G^2(x_A, y_A, x_B, y_B) &\leqq 10, \quad \text{where } G^2(x_A, y_A, x_B, y_B) := y_A + y_B \\ G^3(x_A, y_A, x_B, y_B) &\leqq 0, \quad \text{where } G^2(x_A, y_A, x_B, y_B) := 400 - u_B(x_B, y_B). \end{aligned}$$

Then the first-order marginal conditions are

$$\begin{aligned}
x_A &: \frac{\partial u_A}{\partial x_A} = \sigma_x \frac{\partial G^1}{\partial x_A} + \sigma_y \frac{\partial G^2}{\partial x_A} + \lambda \frac{\partial G^3}{\partial x_A} \\
y_A &: \frac{\partial u_A}{\partial y_A} = \sigma_x \frac{\partial G^1}{\partial y_A} + \sigma_y \frac{\partial G^2}{\partial y_A} + \lambda \frac{\partial G^3}{\partial y_A} \\
x_B &: \frac{\partial u_A}{\partial x_B} = \sigma_x \frac{\partial G^1}{\partial x_B} + \sigma_y \frac{\partial G^2}{\partial x_B} + \lambda \frac{\partial G^3}{\partial x_B} \\
y_B &: \frac{\partial u_A}{\partial y_B} = \sigma_x \frac{\partial G^1}{\partial y_B} + \sigma_y \frac{\partial G^2}{\partial y_B} + \lambda \frac{\partial G^3}{\partial y_B},
\end{aligned} \qquad \text{(FOMC)}$$

in which there's an equation for each variable and the RHS of each equation contains terms corresponding to each of the three constraints. But many of the partial derivatives in these terms have the value 0 or 1, and the equations are therefore much simpler than they at first appear:

$$\begin{aligned}
x_A &: \frac{\partial u_A}{\partial x_A} = \sigma_x \cdot 1 + \sigma_y \cdot 0 + \lambda \cdot 0 \\
y_A &: \frac{\partial u_A}{\partial y_A} = \sigma_x \cdot 0 + \sigma_y \cdot 1 + \lambda \cdot 0 \\
x_B &: 0 = \sigma_x \cdot 1 + \sigma_y \cdot 0 + \lambda(-\frac{\partial u_B}{\partial x_B}) \\
y_B &: 0 = \sigma_x \cdot 0 + \sigma_y \cdot 1 + \lambda(-\frac{\partial u_B}{\partial y_B}),
\end{aligned}$$

which we can write more compactly as

$$\begin{aligned}
x_A &: u_{Ax} = \sigma_x \\
y_A &: u_{Ay} = \sigma_y \\
x_B &: 0 = \sigma_x - \lambda u_{Bx} \\
y_B &: 0 = \sigma_y - \lambda u_{By}.
\end{aligned} \qquad \text{(FOMC)}$$

These are exactly the first four equations in (FOC) above, in (a).

(c) Let's try a different allocation, $((\hat{x}_A, \hat{y}_A), (\hat{x}_B, \hat{y}_B)) = ((15, 5), (10, 5))$, and see if it satisfies (FOC). Now the RHS of the third constraint is 500 instead of 400, and the last three equations are all satisfied again. But now there are no values of σ_x, σ_y, and λ that will satisfy the first four equations, the FOMC. The first two equations require that $\sigma_x = 150$ and $\sigma_y = 100$, but the next two equations require that $\sigma_x = \sigma_y$. Therefore this allocation is not a solution of (P-Max) (and it is therefore not Pareto efficient). It's important to note that this allocation failed only to satisfy the FOMC, the first-order *marginal* conditions. As this suggests, the marginal conditions alone are important: they're necessary for a solution of (P-Max), and if the "complementary slackness" conditions are satisfied, then the marginal conditions are sufficient as well.

(d) So now let's focus on just the marginal conditions. The first two equations in (FOC) in (a), and the second two equations, respectively, yield

$$2\frac{y_A}{x_A} = \frac{\sigma_x}{\sigma_y} \quad \text{and} \quad 2\frac{y_B}{x_B} = \frac{\sigma_x}{\sigma_y} \ ,$$

which in turn yield

$$\frac{y_A}{x_A} = \frac{y_B}{x_B} \ . \tag{1}$$

If an allocation satisfies equation (1) then we can clearly find values of σ_x, σ_y, and λ that will satisfy the four (FOMC) equations, *i.e.*, the first four equations in (FOC). If the allocation also fully allocates the endowment bundle, then it's Pareto efficient. The equation (1) therefore becomes a single marginal condition with which we can check whether *any* interior allocation is Pareto efficient (in this specific two-person, two-good example).

(e) In (d) we obtained equation (1), in which the Lagrange multipliers dropped out, leaving only the variables of the problem (P-Max), the quantities of the goods. Let's use equation (1) to determine all the Pareto allocations. Since we know that a Pareto allocation must fully allocate the endowment bundle $(\mathring{x}, \mathring{y}) = (25, 10)$, we have $x_B = \mathring{x} - x_A$ and $y_B = \mathring{y} - y_A$. So equation (1) can be rewritten as

$$\frac{y_A}{x_A} = \frac{\mathring{y} - y_A}{\mathring{x} - x_A} \ , \qquad \textit{i.e.,} \quad \frac{y_A}{x_A} = \frac{10 - y_A}{25 - x_A} \ , \tag{2}$$

and performing a little bit of arithmetic on (2),

$$\begin{aligned} 25y_A - x_A y_A &= 10x_A - x_A y_A \\ \textit{i.e.,} \qquad 25y_A &= 10x_A \\ \textit{i.e.,} \qquad y_A &= \tfrac{2}{5}x_A. \end{aligned}$$

The whole set of all interior Pareto allocations is now seen to be the straight line from the SW corner of the Edgeworth box to the NE corner. (Note that, from (1), we also have $y_B = \frac{2}{5}x_B$.)

(f) Now notice that the arithmetic we performed above, on equation (2), could have instead been performed on the version that appears on the left in (2), the version that has only $\mathring{x}$ and $\mathring{y}$ and not 25 and 10:

$$\begin{aligned} \mathring{x}y_A - x_A y_A &= \mathring{y}x_A - x_A y_A \\ \textit{i.e.,} \qquad \mathring{x}y_A &= \mathring{y}x_A \\ \textit{i.e.,} \qquad y_A &= \frac{\mathring{y}}{\mathring{x}}x_A. \end{aligned}$$

This tells us that the particular endowment bundle didn't matter for the result that the Pareto allocations are the ones on the diagonal of the Edgeworth box.

(g) In (d), when we obtained the single marginal condition that we labeled as equation (1), we obtained equation (1) from the first four equations in (FOC) in (a) — *i.e.*, from the FOMC for the specific utility functions in this example (which happened to be the same utility function). But we could have done exactly the same thing with the four (FOMC) equations in (b) — let's say the nice, compactly written version of (FOMC). Just as in (d), the first two equations and the remaining two equations yield, respectively,

$$\frac{u_{Ax}}{u_{Ay}} = \frac{\sigma_x}{\sigma_y} \quad \text{and} \quad \frac{u_{Bx}}{u_{By}} = \frac{\sigma_x}{\sigma_y} \ . \tag{3}$$

Since a consumer's MRS between x and y is $\frac{u_x}{u_y}$, equation (3) yields

$$MRS_A = MRS_B \ . \tag{4}$$

We see that the single marginal condition we've obtained holds not just for the specific utility function $u(x, y) = x^2 y$ in this example, but for any differentiable quasiconcave utility functions u_A and u_B.

This example went through much more detail and showed many more steps than one would normally do. The typical way we would proceed would be to write down an appropriate constrained maximization problem, like (P-Max), whose solutions are the Pareto allocations; then obtain the first-order marginal conditions, like the compact version of (FOMC) in (b); and then use these to obtain economically meaningful marginal conditions, like equations (3) and (4) in (g).

Important: Some of the results in this example are special to the example and don't hold in general. For example, the Pareto allocations don't generally lie on the straight line joining two corners of the Edgeworth box. And since we used the Kuhn-Tucker conditions throughout, the results may not hold for utility functions that aren't continuously differentiable or aren't quasiconcave.