Sequences and Convergence in Metric Spaces

Definition: A **sequence** in a set X (a sequence of elements of X) is a function $s : \mathbb{N} \to X$. We usually denote $s(n)$ by s_n, called *the n-th term of s*, and write $\{s_n\}$ for the sequence, or $\{s_1, s_2, \ldots\}$.

See the nice introductory paragraphs about sequences on page 23 of de la Fuente.

By analogy with $\mathbb{R}^n$, we use the notation $\mathbb{R}^\infty$ to denote the set of sequences of real numbers, and we use the notation X^∞ to denote the set of sequences in a set X. (But ∞ is *not* an element of $\mathbb{N}$, a natural number, so this notation is indeed simply an analogy.)

Example: Let V be a vector space. The set V^∞ of all sequences in V is a vector space under the natural component-wise definitions of vector addition and scalar multiplication:

$$\{x_1, x_2, \ldots\} + \{y_1, y_2, \ldots\} := \{x_1 + y_1, x_2 + y_2, \ldots\} \quad \text{and} \quad \alpha\{x_1, x_2, \ldots\} := \{\alpha x_1, \alpha x_2, \ldots\}.$$

Earlier, when we defined $\mathbb{R}^n$ as a vector space, we defined vector addition and scalar multiplication in $\mathbb{R}^n$ component-wise, from the addition and multiplication of the real-number components of the n-tuples in $\mathbb{R}^n$, just as we've done in the example above for V^∞. Unlike $\mathbb{R}^n$, however, the vector spaces $\mathbb{R}^\infty$ and V^∞ are not finite-dimensional. For example, no finite subset forms a basis of the set $\mathbb{R}^\infty$ of sequences of real numbers (*i.e.*, sequences in $\mathbb{R}$).

Because $\mathbb{R}^\infty$ is a vector space, we could potentially define a norm on it. But the norms we defined on $\mathbb{R}^n$ don't generalize in a straightforward way to $\mathbb{R}^\infty$. For example, you should be able to easily provide an example of a sequence for which neither $\max\{|x_1|, |x_2|, \ldots\}$ nor $\sum_{n=1}^{\infty} |x_n|$ is a real number. This is one symptom of the fact that the set of *all* sequences in a space generally doesn't have nice properties. But we're often interested only in sequences that *do* have nice properties — for example, the set of all *bounded* sequences.

Definition: A sequence $\{x_n\}$ of real numbers is **bounded** if there is a number $M \in \mathbb{R}$ for which every term x_n satisfies $|x_n| \leqq M$. More generally, a sequence $\{x_n\}$ in a normed vector space is **bounded** if there is a number $M \in \mathbb{R}$ for which every term x_n satisfies $||x_n|| \leqq M$.

Remark: We use the notation ℓ^∞ for the set of all bounded real sequences, equipped with the norm $||\{x_n\}||_\infty := \sup\{|x_1|, |x_2|, \ldots\}$. Note that this is a subset of $\mathbb{R}^\infty$. Note too that we needed to change *max* to *sup* in the definition of $||\{x_n\}||_\infty$ in order that the norm be well-defined: the sequence $x_n = 1 - (1/n)$, for example, is in ℓ^∞ (it's bounded), and $\sup\{|x_n| \mid n \in \mathbb{N}\} = 1$, but $\max\{|x_n| \mid n \in \mathbb{N}\}$ is not defined.

Exercise: Verify that ℓ^∞ is a vector subspace of $\mathbb{R}^\infty$ and that $||\{x_n\}||_\infty$ is indeed a norm. Therefore ℓ^∞ is a normed vector space.

We can easily convert our definition of bounded sequences in a normed vector space into a definition of bounded sets and bounded functions. And by replacing the norm in the definition with the distance function in a metric space, we can extend these definitions from normed vector spaces to general metric spaces.

Definition: A subset S of a metric space (X, d) is **bounded** if

$$\exists \bar{x} \in X, M \in \mathbb{R} : \forall x \in S : d(x, \bar{x}) \leqq M.$$

A function $f : D \to (X, d)$ is **bounded** if its image $f(D)$ is a bounded set.

Since a sequence in a metric space (X, d) is a function from $\mathbb{N}$ into X, the definition of a bounded function that we've just given yields the result that a sequence $\{x_n\}$ in a metric space (X, d) is bounded if and only if

$$\exists \bar{x} \in X, M \in \mathbb{R} : \forall n \in \mathbb{N} : d(x_n, \bar{x}) \leqq M,$$

so that the definition we gave earlier for a bounded sequence of real numbers is simply a special case of this more general definition.

Remark: Every element of $C[0, 1]$ is a bounded function.

Question: Is $C[0, 1]$ a bounded set, for example under the max norm $||f||_\infty$?

Exercise: Let S be the set of all real sequences that have only a finite number of non-zero terms — *i.e.*, $S = \{\{x_n\} \in \mathbb{R}^\infty \mid x_n \neq 0 \text{ for a finite set } A \subseteq \mathbb{N}\}$. Determine whether S is a vector subspace of ℓ^∞ . If it is, provide a proof; if it isn't, show why not.

Another important set of sequences is the set of *convergent* sequences, which we study next.

Convergence of Sequences

Definition: A sequence $\{x_n\}$ of real numbers **converges** to $\bar{x} \in \mathbb{R}$ if

$$\forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow |x_n - \bar{x}| < \epsilon.$$

Example 1: The sequence $x_n = \dfrac{1}{n}$ converges to 0.

Example 2: The sequence $x_n = (-1)^n$ does not converge.

Example 3: The sequence

$$x_n = \begin{cases} 1, & \text{if } n \text{ is a square, } i.e. \text{ if } n \in \{1, 4, 9, 16, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

does not converge, despite the fact that it has ever longer and longer strings of terms that are zero.

If we replace $|x_n - \bar{x}|$ in this definition with the distance notation $d(x_n, \bar{x})$, then the definition applies to sequences in any metric space:

Definition: Let (X, d) be a metric space. A sequence $\{x_n\}$ in X **converges** to $\bar{x} \in X$ if

$$\forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \epsilon.$$

We say that $\bar{x}$ is the limit of $\{x_n\}$, and we write $\lim\{x_n\} = \bar{x}$, $x_n \to \bar{x}$, and $\{x_n\} \to \bar{x}$.

Example: A convergent sequence in a metric space is bounded; therefore the set of convergent real sequences is a subset of ℓ^∞. You should be able to verify that the set is actually a vector subspace of ℓ^∞. This is not quite as trivial as it might at first appear: you have to show that the set of convergent sequences is closed under vector addition and scalar multiplication — that if two sequences $\{x_n\}$ and $\{y_n\}$ both converge, then the sequences $\{x_n\} + \{y_n\}$ and $\alpha\{x_n\}$ both converge.

Example: For each $n \in \mathbb{N}$, let $a_n = \frac{n}{n+1}$ and define $F_n : [0, 1] \to \mathbb{R}$ by

$$F_n(x) = \begin{cases} \dfrac{1}{a_n} x & , \text{ if } x < a_n \\ 1 & , \text{ if } x \geqq a_n. \end{cases}$$

Then $F_n \to F$ in $C([0, 1])$ with the max-norm, where $F(x) = x$. What is the value of $||F_n - F||$? Note that F_n is the cdf of the uniform distribution on the interval $[0, a_n]$, and $\{F_n\}$ converges to the cdf of the uniform distribution on $[0, 1]$.

Definition: Let (X, d) be a metric space, let $\bar{x} \in X$, and let $r \in \mathbb{R}_{++}$.
(1) The open ball about $\bar{x}$ of radius r is the set $B(\bar{x}, r) := \{x \in X \mid d(x, \bar{x}) < r\}$.
(2) The closed ball about $\bar{x}$ of radius r is the set $\bar{B}(\bar{x}, r) := \{x \in X \mid d(x, \bar{x}) \leqq r\}$.

Remark: Let (X, d) be a metric space. A subset $S \subseteq X$ is bounded if and only if it is contained in an open ball — and equivalently, if and only if it is contained in a closed ball.

Remark: Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to $\bar{x}$ if and only if for every $\epsilon > 0$, "x_n is eventually in $B(\bar{x}, \epsilon)$" — *i.e.*, $\exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow x_n \in B(\bar{x}, \epsilon)$.

Here's a proof that the sum of two convergent sequences in a normed vector space is a convergent sequence:

Proposition: If $\{x_n\}$ and $\{y_n\}$ are sequences in a normed vector space $(V, ||\cdot||)$ and if $\{x_n\} \to \bar{x}$ and $\{y_n\} \to \bar{y}$, then $\{x_n\} + \{y_n\} \to \bar{x} + \bar{y}$.

Proof: Let $\epsilon > 0$. Because $\{x_n\} \to \bar{x}$ and $\{y_n\} \to \bar{y}$, there are $\bar{n}_x$ and $\bar{n}_y$ such that

$$n > \bar{n}_x \Rightarrow ||x_n - \bar{x}|| < \epsilon/2 \quad \text{and} \quad n > \bar{n}_y \Rightarrow ||y_n - \bar{y}|| < \epsilon/2.$$

Let $\bar{n} = \max\{\bar{n}_x, \bar{n}_y\}$; then

$$n > \bar{n} \Rightarrow ||x_n - \bar{x}|| + ||y_n - \bar{y}|| < \epsilon/2 + \epsilon/2 = \epsilon.$$

But we have

$$\begin{aligned} ||(x_n + y_n) - (\bar{x} + \bar{y})|| &= ||(x_n - \bar{x}) + (y_n - \bar{y})|| \\ &\leqq ||x_n - \bar{x}|| + ||y_n - \bar{y}||, \text{ by the Triangle Inequality.} \end{aligned}$$

Therefore

$$n > \bar{n} \Rightarrow ||(x_n + y_n) - (\bar{x} + \bar{y})|| < \epsilon,$$

and since $x_n + y_n$ is the n^{th} term of the sequence $\{x_n\} + \{y_n\}$, this concludes the proof. □

The Least Upper Bound Property
and the Completeness Axiom for $\mathbb{R}$

Definitions: Let S be a subset of $\mathbb{R}$. An **upper bound** of S is a number b such that $x \leqq b$ for every $x \in S$. A **least upper bound** of S is a b^* such that $b^* \leqq b$ for every b that's an upper bound of S.

Remark: In $\mathbb{R}$, a set can have no more than one least upper bound, so it makes sense to talk about *the* least upper bound of S. It is also called the *supremum* of S, denoted $\sup S$ or $\operatorname{lub} S$.

Lower bound, greatest lower bound, glb and inf are defined analogously.

Definition: A partially ordered set X has the **LUB Property** if every nonempty set that has an upper bound has a least upper bound.

The Completeness Axiom: $\mathbb{R}$ has the LUB property — any nonempty set of real numbers that has an upper bound has a least upper bound.

Most sequences, of course, don't converge. Even if we restrict attention to bounded sequences, there is no reason to expect that a bounded sequence converges. Here's a condition that *is* sufficient to ensure that a sequence converges, and it tells us what the limit of the sequence is.

The Monotone Convergence Theorem: Every bounded monotone sequence in $\mathbb{R}$ converges to an element of $\mathbb{R}$.

Proof: Let $\{x_n\}$ be a monotone increasing sequence of real numbers. Since it's bounded, it has a least upper bound b. We will show that $\{x_n\} \to b$. Suppose $\{x_n\}$ doesn't converge to b. Then for some $\epsilon > 0$, infinitely many terms of the sequence satisfy $|x_n - b| \geqq \epsilon$ — *i.e.*, $x_n \leqq b - \epsilon$ (x_n cannot be greater than b if b is an upper bound). It follows that $x_n \leqq b - \epsilon$ for *all* $n \in \mathbb{N}$: since $\{x_n\}$ is increasing, if $x_m > b - \epsilon$ for some m, then $x_m > b - \epsilon$ for all larger n, contradicting that $x_n \leqq b - \epsilon$ for infinitely many n. Thus we have $x_n \leqq b - \epsilon$ for all $n \in \mathbb{N}$; *i.e.*, $b - \epsilon$ is an upper bound of $\{x_1, x_2, \ldots\}$, and therefore b is not a least upper bound of $\{x_1, x_2, \ldots\}$, a contradiction. Therefore $\{x_n\}$ does converge to b.

If $\{x_n\}$ is a monotone decreasing sequence, the above proof shows that the increasing sequence $\{-x_n\}$ converges, and therefore $\{x_n\}$ converges. $\square$

Subsequences and Cluster Points

Definition: Let $f : X \to Y$ be a function and let A be a subset of X. The **restriction** of f to A, denoted $f|_A$, is the function $f|_A : A \to Y$ defined by

$$\forall x \in A : f|_A(x) = f(x).$$

Definition: Let $\{x_n\}$ be a sequence in X — *i.e.*, $x : \mathbb{N} \to X$. A **subsequence** of $\{x_n\}$, denoted $\{x_{n_k}\}$, is the restriction of the function $x(\cdot)$ to an infinite subset of $\mathbb{N}$.

Here is an alternative, equivalent definition:

Definition: Let $\{x_n\}$ be a sequence in X. A **subsequence** of $\{x_n\}$ is the sequence $\{x_{n_k}\}$ for a strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$.

Remark: A subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ in X is also a sequence in X.

Example 1: Let $x_n = n$, and $n_k = k^2$. Then $x_n = \{1, 2, 3, 4, \ldots\}$ and $\{x_{n_k}\} = \{1, 4, 9, 16, \ldots\}$.

Example 2: Let $x_n = (-1)^n$, *i.e.*, $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$.

If $n_k = 2k$, then $\{x_{n_k}\} = \{1, 1, 1, 1, \ldots\}$.
If $n_k = 2k - 1$, then $\{x_{n_k}\} = \{-1, -1, -1, -1, \ldots\}$.
If $n_k = k^2$, then $\{x_{n_k}\} = \{-1, 1, -1, 1, \ldots\}$.

Example 3: Let $x_n = (-1)^n \dfrac{n-1}{n}$, *i.e.*, $\{x_n\} = \{0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots\}$.

If $n_k = 2k$, then $\{x_{n_k}\} = \{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots\}$.
If $n_k = 2k - 1$, then $\{x_{n_k}\} = \{0, -\frac{2}{3}, -\frac{4}{5}, -\frac{6}{7}, \ldots\}$.
If $n_k = k^2$, then $\{x_{n_k}\} = \{0, \frac{3}{4}, -\frac{8}{9}, \frac{15}{16}, -\frac{24}{25}, \ldots\}$.

Definition: Let $\{x_n\}$ be a sequence in a metric space (X, d). A **cluster point** of $\{x_n\}$ is an element $\bar{x} \in X$ such that every open ball around $\bar{x}$ contains an infinite number of terms of $\{x_n\}$ — *i.e.*, such that for every $\epsilon > 0$, the set $\{n \in \mathbb{N} \mid x_n \in B(\bar{x}, \epsilon\}$ is an infinite set. We also say that for every $\epsilon > 0$, "x_n is frequently in $B(\bar{x}, \epsilon\}$."

Remark: If $\{x_n\}$ converges to $\bar{x}$, then $\bar{x}$ is its only cluster point. Therefore, if $\{x_n\}$ has more than one cluster point, it doesn't converge.

Remark: If $\{x_n\}$ converges to $\bar{x}$, then every subsequence of $\{x_n\}$ converges to $\bar{x}$. Therefore, if $\{x_n\}$ has a subsequence that doesn't converge, then $\{x_n\}$ doesn't converge.

Remark: $\bar{x}$ is a cluster point of $\{x_n\}$ if and only if there is a subsequence that converges to $\bar{x}$.

Examples: In Example 1, $\{x_n\}$ does not converge and in fact has no cluster points, so it has no convergent subsequences. In Example 2, $\{x_n\}$ has exactly two cluster points, 1 and -1, so the sequence doesn't converge; we exhibited a subsequence that converges to 1, a subsequence that converges to -1, and a subsequence that doesn't converge. In Example 3, $\{x_n\}$ has the same two cluster points, 1 and -1, and for each one we exhibited a subsequence that converges to it.

Terminology: We've introduced the terminology "$\{x_n\}$ is eventually in $B(\bar{x}, \epsilon\}$" and "$\{x_n\}$ is frequently in $B(\bar{x}, \epsilon\}$." More generally, for any property P that a sequence might have, we say that "$\{x_n\}$ eventually has Property P" if $\exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow x_n$ has Property P, and that "$\{x_n\}$ frequently has Property P" if $\{x_n\}$ has Property P for all n in an infinite subset of $\mathbb{N}$ — *i.e.*, if some subsequence of $\{x_n\}$ has Property P. Clearly, a sequence eventually has a property P if and only if the sequence does not frequently have Property $\sim P$ (the negation of P); and a sequence frequently has Property P if and only if it does not eventually have Property $\sim P$.