# The Solution Function and the Value Function for a Maximization Problem

Consider the maximization problem

$$\max_{x \in X} f(x;\theta) \ \text{ subject to } \ G(x;\theta) \leqq \mathbf{0} \tag{$\mathbf{P}$}$$

for values of $\theta$ in some set $\Theta$. Note that we're maximizing over $x$ and not over $\theta$: $x$ is a variable in the problem (typically a vector or $n$-tuple of variables) and $\theta$ is a parameter (typically a vector or $m$-tuple of parameters). The parameters may appear in the objective function and/or the constraints, if there are any constraints. We associate the following two functions with the maximization problem ($\mathbf{P}$), where we're assuming that for each $\theta \in \Theta$ the problem ($\mathbf{P}$) has a unique solution:

the **solution function**: $x(\theta)$ is the $x$ that's the solution of ($\mathbf{P}$)

the **value function**: $v(\theta) := f(x(\theta), \theta)$ .

The solution function $x : \Theta \to X$ gives the solution $x$ as a function of the parameters; the value function $v : \Theta \to \mathbb{R}$ gives the value of the objective function as a function of the parameters.

**Example 1:** The consumer maximization problem (CMP) in demand theory,

$$\max_{\mathbf{x} \in \mathbb{R}^\ell_+} u(\mathbf{x}) \ \text{ subject to } \ \mathbf{p} \cdot \mathbf{x} \leqq w$$

Here $\theta$ is the $(\ell + 1)$-tuple $(\mathbf{p}; w)$ consisting of the price-list $\mathbf{p}$ and the consumer's wealth $w$.

The solution function is the consumer's demand function $\mathbf{x}(\mathbf{p}; w)$ .

The value function is the consumer's indirect utility function $v(\mathbf{p}; w) = u(\mathbf{x}(\mathbf{p}; w))$ .

**Example 2:** The expenditure-minimization problem (EMP) in demand theory,

$$\min_{\mathbf{x} \in \mathbb{R}^\ell_+} E(\mathbf{x}; \mathbf{p}) = \mathbf{p} \cdot \mathbf{x} \ \text{ subject to } \ u(\mathbf{x}) \geqq \overline{u}.$$

Here $\theta$ is the $(\ell + 1)$-tuple $(\mathbf{p}; \overline{u})$ consisting of the price-list $\mathbf{p}$ and the consumer's target level of utility, $\overline{u}$.

The solution function is the consumer's Hicksian (compensated) demand function $h(\mathbf{p}, \overline{u})$.

The value function is the consumer's expenditure function $e(\mathbf{p}, \overline{u}) = E\big(h(\mathbf{p}, \overline{u}), \mathbf{p}\big)$ .

**Example 3:** The firm's cost-minimization (*i.e.*, expenditure-minimization) problem,

$$\min_{\mathbf{x}\in\mathbb{R}^\ell_+} E(\mathbf{x};\mathbf{w}) = \mathbf{w}\cdot\mathbf{x} \;\text{ subject to }\; F(\mathbf{x}) \geqq y.$$

Here $F$ is the firm's production function; $\mathbf{x}$ is the $\ell$-tuple of input levels that will be employed; $E(\mathbf{x};\mathbf{w})$ is the resulting expenditure the firm will incur; and $\theta$ is the $(\ell+1)$-tuple $(y;\mathbf{w})$ consisting of the proposed level of output, $y$, and the $\ell$-tuple $\mathbf{w}$ of input prices.

The solution function is the firm's input demand function $\mathbf{x}(y;\mathbf{w})$ .

The value function is the firm's cost function $C(y;\mathbf{w}) = E(\mathbf{x}(y;\mathbf{w});\mathbf{w})$ .

**Example 4:** The Pareto problem (P-Max),

$$\max_{\mathbf{x}\in\mathcal{F}} u^1(\mathbf{x}^1) \;\text{ subject to }\; u^2(\mathbf{x}^2) \geqq u_2\,,\;\ldots\;,\; u^n(\mathbf{x}^n) \geqq u_n,$$

where $\mathcal{F}$ is the feasible set $\{\mathbf{x}\in\mathbb{R}^{n\ell}_+ \mid \sum_1^n \mathbf{x}^i \leqq \mathring{\mathbf{x}}\}$. (Note that we're using superscripts for utility *functions* and subscripts for utility *levels.*) Here $\theta$ is the $(n-1)$-tuple of utility levels $u_2,\ldots,u_n$.

The solution function is $\mathbf{x}(u_2,\ldots,u_n)$, which gives the Pareto allocation as a function of the utility levels $u_2,\ldots,u_n$ .

The value function is $u^1(\mathbf{x}(u_2,\ldots,u_n))$ , which gives the maximum attainable utility level $u_1$ as a function of the utility levels $u_2,\ldots,u_n$ .

The value function therefore describes the utility frontier for the economy $((u^i)_1^n,\mathring{\mathbf{x}})$, as depicted in the diagram below for the case $n=2$.