Economics 519 Final Exam Solutions
Fall 2015

1. Let ϵ be a positive real number and let (X, d) be a metric space in which the metric d satisfies the condition $x' \neq x \Rightarrow d(x, x') \geqq \epsilon$.

(a) Which sequences in X converge and which sequences don't converge? Prove that your answer is correct.

Solution: A sequence $\{x_n\}$ converges if and only if the sequence is eventually constant — *i.e.*, if and only if there is an $\overline{n} \in \mathbb{N}$ such that $n > \overline{n} \Rightarrow x_n = x_{\overline{n}}$. **Proof:** If the sequence is eventually constant, it obviously converges, to $x_{\overline{n}}$. Conversely, suppose $\{x_n\}$ converges to $\overline{x}$. Then $\exists \overline{n} : n > \overline{n} \Rightarrow d(x_n, \overline{x}) < \epsilon$; but $d(x_n, \overline{x}) < \epsilon \Rightarrow x_n = \overline{x}$, so we have $n > \overline{n} \Rightarrow x_n = \overline{x}$. ∥

(b) For which sets X is such a metric space (X, d) compact, and for which sets X is such a space not compact? Prove that your answer is correct. ("Compact" here means that the set has the Bolzano-Weierstrass Property.)

Solution: The compact subsets of X are the finite subsets. **Proof:** If X is finite, then every sequence in X must have an infinite number of terms that are identical — *i.e.*, there must be an $\overline{x} \in X$ such that the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that satisfies $x_{n_k} = \overline{x}$ for all $k \in \mathbb{N}$, and therefore $\{x_{n_k}\}$ converges to $\overline{x}$. In other words, X has the B-W Property. Conversely, if X is infinite, then we can define a sequence $\{x_n\}$ for which each term is a distinct element of X; therefore no subsequence is eventually constant, so $\{x_n\}$ has no convergent subsequence, and X therefore does not have the B-W Property. ∥

2. Let X_1 and X_2 be sets in $\mathbb{R}^n$; let $\widehat{\mathbf{x}}_1 \in X_1$ and $\widehat{\mathbf{x}}_2 \in X_2$; let $\widehat{\mathbf{x}} = \widehat{\mathbf{x}}_1 + \widehat{\mathbf{x}}_2$; and let $\mathbf{p} \neq \mathbf{0} \in \mathbb{R}^n$. Prove that $\widehat{\mathbf{x}}$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on $X_1 + X_2$ if and only if $\widehat{\mathbf{x}}_1$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on X_1 and $\widehat{\mathbf{x}}_2$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on X_2.

Proof: Assume that $\widehat{\mathbf{x}}_i$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on X_i $(i = 1, 2)$. Let $\mathbf{x} \in X_1 + X_2$; then there exist $\mathbf{x}_1 \in X_1$ and $\mathbf{x}_2 \in X_2$ such that $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}$, and therefore $\mathbf{p} \cdot \mathbf{x}_i \leqq \mathbf{p} \cdot \widehat{\mathbf{x}}_i$ $(i = 1, 2)$. Therefore

$$\mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{p} \cdot \mathbf{x}_1 + \mathbf{p} \cdot \mathbf{x}_2 \leqq \mathbf{p} \cdot \widehat{\mathbf{x}}_1 + \mathbf{p} \cdot \widehat{\mathbf{x}}_2 = \mathbf{p} \cdot \widehat{\mathbf{x}}.$$

— *i.e.*, $\widehat{\mathbf{x}}$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on $X_1 + X_2$. Conversely, assume that $\widehat{\mathbf{x}}$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on $X_1 + X_2$, and without loss of generality suppose that $\widehat{\mathbf{x}}_1$ does not maximize $\mathbf{p} \cdot \mathbf{x}$ on X_1. Then for some $\mathbf{x}_1 \in X_1$ we have $\mathbf{p} \cdot \mathbf{x}_1 > \mathbf{p} \cdot \widehat{\mathbf{x}}_1$, and therefore $\mathbf{x}_1 + \widehat{\mathbf{x}}_2 \in X_1 + X_2$ and

$$\mathbf{p} \cdot (\mathbf{x}_1 + \widehat{\mathbf{x}}_2) = \mathbf{p} \cdot \mathbf{x}_1 + \mathbf{p} \cdot \widehat{\mathbf{x}}_2 > \mathbf{p} \cdot \widehat{\mathbf{x}}_1 + \mathbf{p} \cdot \widehat{\mathbf{x}}_2 = \mathbf{p} \cdot (\widehat{\mathbf{x}}_1 + \widehat{\mathbf{x}}_2) = \mathbf{p} \cdot \widehat{x},$$

so $\widehat{\mathbf{x}}$ does not maximize $\mathbf{p} \cdot \mathbf{x}$ on $X_1 + X_2$, a contradiction. Therefore $\widehat{\mathbf{x}}_1$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on X_1 and $\widehat{\mathbf{x}}_2$ maximizes $\mathbf{p} \cdot \mathbf{x}$ on X_2. ∥

3. Let S be a convex subset of $\mathbb{R}^n$; let $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ be concave functions; and let $f : S \to \mathbb{R}$ be defined by $\forall x \in S : f(x) = \min\{f_1(x), f_2(x)\}$. Prove that f is concave.

Proof: Let $\mathbf{x}_a, \mathbf{x}_b \in S$, let $\lambda \in (0,1) \subset \mathbb{R}$, and let $\mathbf{x} = (1-\lambda)\mathbf{x}_a + \lambda\mathbf{x}_b$. Because each f_i is concave and for all $\mathbf{z} \in S : f_i(\mathbf{z}) \geqq f(\mathbf{z})$ for $i = 1, 2$, we have

$$f_i(\mathbf{x}) \geqq (1-\lambda)f_i(\mathbf{x}_a) + \lambda f_i(\mathbf{x}_b) \geqq (1-\lambda)f(\mathbf{x}_a) + \lambda f(\mathbf{x}_b) \text{ for } i = 1, 2.$$

We also have either $f(\mathbf{x}) = f_1(\mathbf{x})$ or $f(\mathbf{x}) = f_2(\mathbf{x})$, and the inequalities above therefore yield $f(\mathbf{x}) \geqq (1-\lambda)f(\mathbf{x}_a) + \lambda f(\mathbf{x}_b)$. ‖

Open-book part: Be sure to turn in your solutions to Problems #1 - #3 before using notes.

4. Provide an example of a sequence $\{f_n\}$ of continuous real functions defined on the unit interval $[0,1]$ — *i.e.*, functions in $C([0,1])$ — that converges pointwise to a continuous function f in $C([0,1])$ but does not converge uniformly — *i.e.*, the sequence does not converge in the normed vector space $C([0,1])$ with the sup-norm. Prove that indeed your sequence does converge pointwise and does not converge uniformly.

Solution: Here's one example: for each $n \in \mathbb{N}$ let $f_n : [0,1] \to \mathbb{R}$, be defined by

$$f_n(x) = \begin{cases} nx, & x \leqq \frac{1}{2n} \\ 1 - nx, & \frac{1}{2n} \leqq x \leqq \frac{1}{n} \\ 0, & x \geqq \frac{1}{n}. \end{cases}$$

You should draw one or two of the functions f_n. For each $x \in [0,1]$ the sequence $\{f_n(x)\}$ of real numbers is eventually zero, so the sequence obviously converges pointwise to the function $f(x) \equiv 0$. But for any $\epsilon < \frac{1}{2}$, no term of the sequence f_n is within ϵ of f — *i.e.*, for every n there are values of x for which $|f_n(x) - f(x)| > \epsilon$. Therefore $\{f_n\}$ does not converge uniformly. ‖

5. Provide a proof by induction that between any two rational numbers there are infinitely many rational numbers — *i.e.*, that if a and b are rational numbers then for every $n \in \mathbb{N}$ there are n rational numbers x that satisfy $a < x < b$.

Proof: We first show that the conclusion is true for $n = 1$ — *i.e.*, there is a rational number between a and b. In particular, let $x = \frac{1}{2}(a + b)$. Clearly $a < x < b$, and x is rational, as follows: Since a and b are rational, they can be expressed as $a = k_a/m_a$ and $b = k_b/m_b$ for some integers k_a, k_b, m_a, and m_b. We therefore have

$$x = \frac{k_a}{2m_a} + \frac{k_b}{2m_b} = \frac{2k_a m_b + 2k_b m_a}{4m_a m_b},$$

which is also a ratio of integers, so x is rational.

Now assume that the conclusion is true for n (the induction hypothesis) — *i.e.*, there are distinct rational numbers $x_1, \ldots, x_n$ that all satisfy $a < x_i < b$. Wlog suppose that x_n is the largest of these. Then, as above, there is a rational number x_{n+1} that satisfies $x_n < x_{n+1} < b$; and since x_n is larger than the other x_i, the new rational number x_{n+1} is not the same as any of our first n rationals x_i. ‖

6. Let (X, d_X) and (Y, d_Y) be metric spaces; let $F_1 : X \twoheadrightarrow Y$ and $F_2 : X \twoheadrightarrow Y$ be correspondences; and let $F : X \twoheadrightarrow Y$ be the correspondence defined by $\forall x \in X : F(x) = F_1(x) \cup F_2(x)$. Prove that if F_1 and F_2 are UHC then F is UHC.

Proof: Let $\overline{x} \in X$; we will show that F is UHC at $\overline{x}$. Let V be an open subset of Y for which $F(\overline{x}) \subseteqq V$. We'll show that there is an open set $U \subseteqq X$ for which $\overline{x} \in U$ and $x \in U \Rightarrow F(x) \subseteqq V$. Since F_1 and F_2 are both UHC, there are open sets $U_1, U_2 \subseteqq X$ such that $\overline{x} \in U_1$, $\overline{x} \in U_2$, and $x \in U_i \Rightarrow F_i(x) \subseteqq V$ $(i = 1, 2)$. Now let $U = U_1 \cap U_2$; then U is open, and we have $\overline{x} \in U$. Moreover, for any $x \in U$ we have $x \in U_1$, so $F_1(x) \subseteqq V$, and $x \in U_2$, so $F_2(x) \subseteqq V$. Therefore $F_1(x) \cup F_2(x) \subseteqq V$ — *i.e.*, $F(x) \subseteqq V$. ‖