

Economics 519 Final Exam Solutions
University of Arizona
Fall 2016

1. Let (X, d) be a metric space (which may or may not be complete) and let $f : X \rightarrow X$ be a contraction mapping on X . Prove that f has at most one fixed point — *i.e.*, any fixed point must be unique.

Solution: Let β be a contraction modulus of f , and let x and x' be fixed points of f . We will show that $x = x'$. We have $0 < \beta < 1$ and

$$d(x, x') = d(f(x), f(x')) \leq \beta d(x, x').$$

We can have $d(x, x') \leq \beta d(x, x')$ if and only if $d(x, x') = 0$, which in turn is true if and only if $x = x'$. \parallel

2. Prove that if a sequence $\{x_n\}$ converges to \bar{x} in a metric space (X, d) , then every subsequence of $\{x_n\}$ converges to \bar{x} .

Solution: Let $\{x_n\}$ be a sequence that converges to \bar{x} , and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Let $\epsilon > 0$; we must find a $K \in \mathbb{N}$ for which $k > K \Rightarrow d(x_{n_k}, \bar{x}) < \epsilon$. Let \bar{n} be such that $n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \epsilon$, and let K be such that $n_K \geq \bar{n}$. Then $k > K \Rightarrow n_k > \bar{n}$, and therefore $k > K \Rightarrow d(x_{n_k}, \bar{x}) < \epsilon$. \parallel

3. Define the relation \succsim on \mathbb{R}^2 by $\mathbf{x}' \succsim \mathbf{x} \Leftrightarrow [x'_1 \geq x_1 \ \& \ x'_2 \geq x_2]$, and define the strict relation \succ and the equivalence relation \sim from \succsim in the usual way.

(a) In a diagram, depict the weak upper- and lower-contour sets of a typical point $\bar{x} \in \mathbb{R}^2$.

Solution: See Figure 1. \parallel

(b) Prove that \succsim is a preorder, *i.e.*, is reflexive and transitive.

Solution:

Proof that \succsim is reflexive: $x_1 \geq x_1$ and $x_2 \geq x_2$ for every $\mathbf{x} \in \mathbb{R}^2$; therefore $\mathbf{x} \succsim \mathbf{x}$.

Proof that \succsim is transitive: Assume that $\mathbf{x} \succsim \mathbf{x}'$ and $\mathbf{x}' \succsim \mathbf{x}''$. Then $x_i \geq x'_i$ and $x'_i \geq x''_i$ for $i = 1, 2$. Therefore $x_i \geq x''_i$ for $i = 1, 2$, by transitivity in \mathbb{R} , so we have $\mathbf{x} \succsim \mathbf{x}''$. \parallel

(c) Is \succsim a complete preorder? What is the equivalence class $[\bar{x}]$ of a typical $\bar{x} \in \mathbb{R}^2$? Verify your answers.

Solution: \succsim is not complete: if $x_1 > x'_1$ and $x_2 < x'_2$ (*i.e.*, if \mathbf{x} lies to the southeast of \mathbf{x}' in Figure 1), then neither $\mathbf{x} \succsim \mathbf{x}'$ nor $\mathbf{x}' \succsim \mathbf{x}$ holds. The equivalence class $[\mathbf{x}]$ of any point

\mathbf{x} is the singleton $\{\mathbf{x}\}$: if $\mathbf{x}' \sim \mathbf{x}$, then $x'_i \geq x_i$ and $x_i \geq x'_i$ for $i = 1, 2$ — *i.e.*, $x'_i = x_i$ for $i = 1, 2$, so $\mathbf{x}' = \mathbf{x}$. \parallel

(d) Let $c \in \mathbb{R}$ be an arbitrary real number, and let $X = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 \leq c\}$. Is a maximal element for \succsim on X also a maximum element? If yes, provide a proof; if not, provide an example of a maximal element that is not a maximum.

Solution: Let \mathbf{x} be any point that satisfies $x_1 + x_2 = c$. Then \mathbf{x} is maximal in X (*i.e.*, there is no $\mathbf{x}' \in X$ that satisfies $\mathbf{x}' \succ \mathbf{x}$), because if $\mathbf{x}' \succ \mathbf{x}$, then $x'_i \geq x_i$ for $i = 1, 2$ and either $x'_1 > x_1$ or $x'_2 > x_2$, and therefore $x'_1 + x'_2 > x_1 + x_2 = c$ — *i.e.*, $\mathbf{x}' \notin X$. But \mathbf{x} is not a maximum on X (*i.e.*, it's not true that $\mathbf{x} \succsim \mathbf{x}'$ for all $\mathbf{x}' \in X$): let $z \neq 0 \in \mathbb{R}$ and let $x'_1 = x_1 + z$ and $x'_2 = x_2 - z$; then $x'_1 + x'_2 = c$, so $\mathbf{x}' \in X$ but $\mathbf{x} \not\succeq \mathbf{x}'$, as we saw in (c). \parallel

4. Prove that the interior of a convex set in \mathbb{R}^n is convex.

Solution: Let S be a convex set; let $\mathbf{x}_1, \mathbf{x}_2 \in \text{int } S$; let $\lambda \in (0, 1)$; and let $\mathbf{x} = (1 - \lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2$. We'll show that $\mathbf{x} \in \text{int } S$. Since $\mathbf{x}_1, \mathbf{x}_2 \in \text{int } S$, there are open balls $B(\mathbf{x}_1, \epsilon_1) \subseteq S$ and $B(\mathbf{x}_2, \epsilon_2) \subseteq S$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. We'll show that $B(\mathbf{x}, \epsilon) \subseteq S$, which will establish that $\mathbf{x} \in \text{int } S$. Let $\mathbf{z} \in B(\mathbf{x}, \epsilon)$, and we'll show that $\mathbf{z} \in S$. Define \mathbf{z}_1 and \mathbf{z}_2 by $\mathbf{z}_1 = \mathbf{x}_1 + \mathbf{z} - \mathbf{x}$ and $\mathbf{z}_2 = \mathbf{x}_2 + \mathbf{z} - \mathbf{x}$, so that $\mathbf{z}_i - \mathbf{x}_i = \mathbf{z} - \mathbf{x}$ for $i = 1, 2$. See Figure 2. Then we have $d(\mathbf{z}_i, \mathbf{x}_i) = d(\mathbf{z}, \mathbf{x}) < \epsilon \leq \epsilon_i$ for $i = 1, 2$, *i.e.*, $\mathbf{z}_i \in B(\mathbf{x}_i, \epsilon_i) \subseteq S$ for $i = 1, 2$. Moreover,

$$\begin{aligned} (1 - \lambda)\mathbf{z}_1 + \lambda\mathbf{z}_2 &= (1 - \lambda)(\mathbf{x}_1 + \mathbf{z} - \mathbf{x}) + \lambda(\mathbf{x}_2 + \mathbf{z} - \mathbf{x}) \\ &= (1 - \lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2 + \mathbf{z} - \mathbf{x} \\ &= \mathbf{x} + \mathbf{z} - \mathbf{x} \\ &= \mathbf{z}; \end{aligned}$$

that is, \mathbf{z} is a convex combination of \mathbf{z}_1 and \mathbf{z}_2 , each of which is in S . Therefore $\mathbf{z} \in S$. \parallel

5. Assume that the sum of any two convex sets in \mathbb{R}^n is convex, and prove by induction that the sum of any finite collection of convex sets in \mathbb{R}^n is convex.

Solution: We wish to prove that if the sets S_1, \dots, S_m are each convex, then the set $\sum_1^m S_i$ is convex. We know that the result is true for $m = 2$ (that was given). The induction hypothesis is that it's true for $m - 1$, and we'll show that it's true for m . We have $\sum_1^m S_i = \sum_1^{m-1} S_i + S_m$. According to the induction hypothesis, we know that the first term in the sum is convex, and we're given that the sum of any two convex sets is convex; therefore $\sum_1^{m-1} S_i + S_m$ is convex — *i.e.*, $\sum_1^m S_i$ is convex. \parallel

6. Let S be the unit simplex in \mathbb{R}_+^2 : $S = \{\mathbf{p} \in \mathbb{R}_+^2 \mid p_1 + p_2 = 1\}$. For some fixed $\dot{\mathbf{x}} \in \mathbb{R}_+^2$, define the budget set correspondence $f : S \rightarrow \mathbb{R}_+^2$ in the usual way:

$$f(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}_+^2 \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \dot{\mathbf{x}}\}.$$

(a) Prove that f is a closed correspondence, *i.e.*, that f has a closed graph. For this you may use, without proving them, such properties of real sequences as these:

$$\text{If } \lim x_n = \bar{x} \text{ and } \lim y_n = \bar{y}, \text{ then } \lim(x_n + y_n) = \bar{x} + \bar{y} \text{ and } \lim x_n y_n = \bar{x} \bar{y}.$$

$$\text{If } \lim x_n = \bar{x} \text{ and } \forall n \in \mathbb{N} : x_n \leq a, \text{ then } \bar{x} \leq a.$$

Solution: Let G denote the graph of f ; we need to show that G is a closed set in the space $S \times \mathbb{R}_+^2$. Let $\{(\mathbf{p}_n, \mathbf{x}_n)\}$ be a sequence in G that converges to the point $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$; we need to show that $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in G$ — *i.e.*, that $\bar{\mathbf{x}} \in f(\bar{\mathbf{p}})$, *i.e.*, that $\bar{\mathbf{p}} \cdot \bar{\mathbf{x}} - \bar{\mathbf{p}} \cdot \dot{\mathbf{x}} \leq 0$. For each $n \in \mathbb{N}$, define $\beta_n = \mathbf{p}_n \cdot \mathbf{x}_n - \mathbf{p}_n \cdot \dot{\mathbf{x}} = p_{n1}x_{n1} + p_{n2}x_{n2} - p_{n1}\dot{x}_1 + p_{n2}\dot{x}_2$. Note that $\beta_n \leq 0$ for each $n \in \mathbb{N}$, because $(\mathbf{p}_n, \mathbf{x}_n) \in G$ for each $n \in \mathbb{N}$. We have

$$\begin{aligned} \bar{\mathbf{p}} \cdot \bar{\mathbf{x}} - \bar{\mathbf{p}} \cdot \dot{\mathbf{x}} &= \bar{p}_1 \bar{x}_1 + \bar{p}_2 \bar{x}_2 - \bar{p}_1 \dot{x}_1 - \bar{p}_2 \dot{x}_2 \\ &= (\lim p_{n1})(\lim x_{n1}) + (\lim p_{n2})(\lim x_{n2}) - (\lim p_{n1})\dot{x}_1 + (\lim p_{n2})\dot{x}_2 \\ &= \lim(p_{n1}x_{n1}) + \lim(p_{n2}x_{n2}) - \lim(p_{n1})\dot{x}_1 - \lim(p_{n2})\dot{x}_2 \\ &= \lim(p_{n1}x_{n1} + p_{n2}x_{n2} - p_{n1}\dot{x}_1 + p_{n2}\dot{x}_2) \\ &= \lim \beta_n \\ &\leq 0, \text{ because } \beta_n \leq 0 \text{ for all } n \in \mathbb{N}. \end{aligned}$$

(b) Now assume that $\beta > \max\{\dot{x}_1, \dot{x}_2\}$ and that $K = \{\mathbf{x} \in \mathbb{R}_+^2 \mid 0 \leq x_1 \leq \beta, 0 \leq x_2 \leq \beta\}$. Define $g : S \rightarrow K$ by $g(\mathbf{p}) = f(\mathbf{p}) \cap K$. Using any results from the lecture notes that you need, prove that g is upper-hemicontinuous.

Solution: The target space of g is the compact set K , and we have a theorem that says that if a correspondence with a compact target space has a closed graph, then the correspondence is UHC. So we need only to show that g has a closed graph. The graph of g is $G \cap (S \times K)$; we've established in (a) that G is closed; $S \times K$ is closed as well; so the graph of g is the intersection of two closed sets, and is therefore closed. \parallel

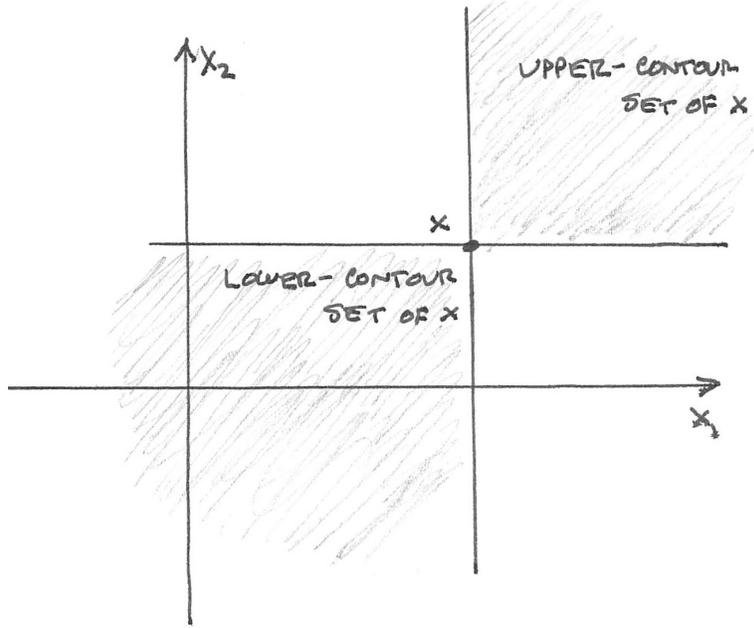


Figure 1

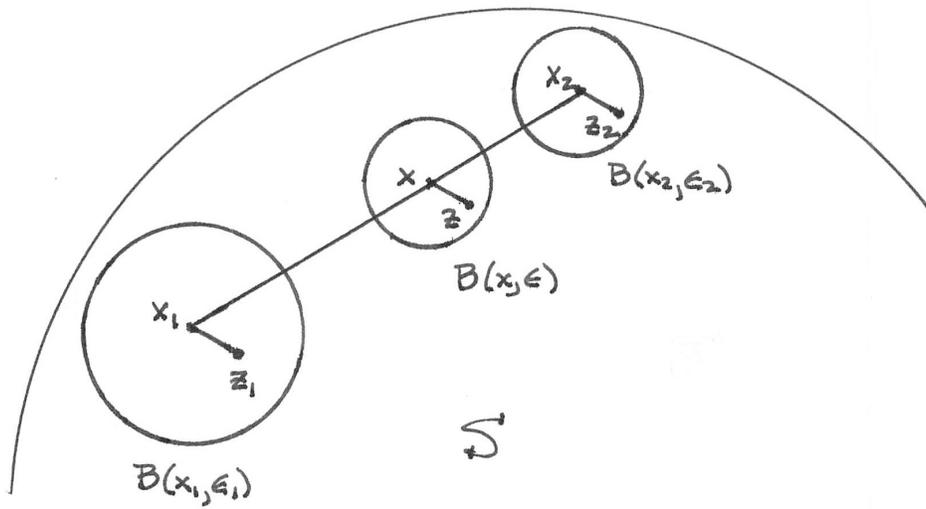


Figure 2