Economics 519 Final Exam Solutions
Fall 2017

1.(a) Prove that if $||\cdot||$ is a norm on a vector space V, then the function $d : V \times V \to \mathbb{R}$ defined by $d(x, x') := ||x - x'||$ is a metric on V.

Solution: Let $x, x', x'' \in V$.

(D1) $d(x, x') = ||x - x'|| \geqq 0$, from (N1).

(D2) $d(x, x') = ||x - x'|| = 0 \Leftrightarrow x - x' = \mathbf{0}$ ($i.e.$, $x = x'$, from (N2)

(D3) $d(x, x') = ||x - x'|| = ||x' - x||$, from commutativity of vector addition; therefore $d(x, x') = d(x', x)$.

(D4) $d(x, x'') = ||x - x' + x' - x''|| \leqq ||x - x'|| + ||x' - x''|$, from (N3); therefore $d(x, x'') \leqq d(x, x') + d(x', x'')$.

1.(b) Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the discrete metric on $\mathbb{R}^n$. If there exists a norm $||\cdot||$ on $\mathbb{R}^n$ for which $d(\mathbf{x}, \mathbf{x}') = ||\mathbf{x} - \mathbf{x}'||$, identify any such norm. If there is no such norm, verify that.

Solution: There is no norm that "generates" the discrete metric in the sense that $d(\mathbf{x}, \mathbf{x}') = ||\mathbf{x} - \mathbf{x}'||$ for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, because (N4) cannot be satisfied by any such function $||\cdot||$. Suppose $||\cdot||$ were such a norm, and let $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$ and $\alpha > 1 \in \mathbb{R}$. Then

$$d(\alpha\mathbf{x}, \mathbf{0}) = ||\alpha\mathbf{x} - \mathbf{0}|| = ||\alpha\mathbf{x}|| = |\alpha|||\mathbf{x}|| > ||\mathbf{x}|| = d(\mathbf{x}, \mathbf{0}).$$

which contradicts the fact that $d(\alpha\mathbf{x}, \mathbf{0}) = d(\mathbf{x}, \mathbf{0})$. More generally, note that for any function $||\cdot||$ that satisfies (N4), the function's range is all of $\mathbb{R}_+$; so if $d(\cdot, \cdot)$ is the metric generated by $||\cdot||$, then the range of d must also be all of $\mathbb{R}_+$. The discrete metric has only a two-element range. (There is one vector space to which this argument doesn't apply: the one-point vector space consisting of only the origin $\mathbf{0}$, for which $d(\mathbf{0}, \mathbf{0}) = ||\mathbf{0} - \mathbf{0}|| = 0$. But for every $n \in \mathbb{N}$, $\mathbb{R}^n$ is not that vector space.)

2. A real number x is rational if it can be expressed as the ratio of two integers — *i.e.*, if $x = k/m$ for some $k, m \in \mathbb{Z}$. Prove by induction that for any $n \in \mathbb{N}$, the sum of n rational numbers is rational. (You may use the fact that $\mathbb{Z}$, the set of integers, is closed under the operations of addition and multiplication.)

Solution: We first show that the conclusion holds for the sum of two rational numbers — *i.e.*, it holds for $n = 2$. Let $x_1, x_2 \in \mathbb{Q}$ and let $k_1, k_2, m_1, m_2 \in \mathbb{Z}$ be such that $x_1 = k_1/m_1$ and $x_2 = k_2/m_2$. (Note that therefore $m_1, m_2 \neq 0$). Then

$$x_1 + x_2 = \frac{k_1 m_2 + k_2 m_1}{m_1 m_2}$$

Since $\mathbb{Z}$ is closed under addition and multiplication, the numerator and denominator of this fraction are both integers (and $m_1 m_2 \neq 0$), and $x_1 + x_2$ is therefore rational.

The induction hypothesis is that the sum of $n-1$ rational numbers $x_1, \ldots, x_{n-1}$ is rational; we show that then the sum of n rationals is rational. Let $x_1, \ldots, x_n$ all be rational numbers and let $s = x_1 + \cdots + x_{n-1}$, which is rational according to the induction hypothesis. We have $x_1 + \cdots + x_n = s + x_n$, which is the sum of two rational numbers, which, as we have shown, is itself a rational number, completing the proof.

Note that this result can be proved directly, *i.e.*, not via induction, by mimicking the proof we gave for $n = 2$. But that proof for general n is kind of ugly.

3. Assume that X is a convex subset of a vector space V and that $f : X \to \mathbb{R}$ is a concave function. Let S be the set of solutions of the constrained maximization problem

$$\max f(x) \text{ subject to } x \in X.$$

Must the set S be convex? If so, prove it; if not, provide a counterexample.

Solution: The set S is convex, as follows. Denote the maximization problem as (P). Suppose $x', x'' \in S$ — *i.e.*, each is a solution of (P) — and let $\widehat{x} = (1-\lambda)x' + \lambda x''$, where $0 < \lambda < 1$. We show that $\widehat{x}$ is a solution of (P). For all $x \in X$ we have both $f(x) \leqq f(x')$ and $f(x) \leqq f(x'')$. Therefore, for all $x \in X$ we have

$$(1-\lambda)f(x) \leqq (1-\lambda)f(x') \text{ and } \lambda f(x) \leqq \lambda f(x''),$$

and therefore

$$(1-\lambda)f(x) + \lambda f(x) \leqq (1-\lambda)f(x') + \lambda f(x''),$$

i.e., $f(x) \leqq (1-\lambda)f(x') + \lambda f(x'')$. But $(1-\lambda)f(x') + \lambda f(x'') \leqq f(\widehat{x})$, because f is concave, so we have $f(x) \leqq f(\widehat{x})$ for all $x \in X$ — *i.e.*, $\widehat{x}$ is a solution of (P), *i.e.*, $\widehat{x} \in S$.

4. A relation R on a set $\mathcal{Z}$ is a **partial order** or a **partial ordering** of $\mathcal{Z}$ if R is reflexive, transitive, and antisymmetric. Two examples of partial orders that we've encountered are the vector partial order on $\mathbb{R}^n$ and the subset partial order $\subseteq$ on the set 2^X of all subsets of a set X. (In the second case, 2^X is playing the role of $\mathcal{Z}$.) Now let X be a set and let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of X; $\mathcal{P}'$ is said to be a **refinement** of $\mathcal{P}$, denoted $\mathcal{P}' \leqq \mathcal{P}$, if for every $E' \in \mathcal{P}'$ there is an $E \in \mathcal{P}$ such that $E' \subseteq E$. We also say that $\mathcal{P}'$ is **at least as informative** as $\mathcal{P}$. Prove that the refinement relation $\leqq$ is a partial ordering of the set of all partitions of X. (It could be helpful to note that if $\sim$ and $\sim'$ are the equivalence relations defined by $\mathcal{P}$ and $\mathcal{P}'$, then $\mathcal{P}' \leqq \mathcal{P}$ if and only if $x \sim' y \Rightarrow x \sim y$ for all $x, y \in X$.)

Solution:

$\leqq$ is reflexive, *i.e.*, $\mathcal{P} \leqq \mathcal{P}$ for every partition $\mathcal{P}$: For any $E \in \mathcal{P}$, we have $E \subseteq E$.

$\leqq$ is transitive, *i.e.*, $[\mathcal{P}'' \leqq \mathcal{P}' \ \& \ \mathcal{P}' \leqq \mathcal{P}] \Rightarrow \mathcal{P}'' \leqq \mathcal{P}$: Assume that $\mathcal{P}'' \leqq \mathcal{P}'$ and $\mathcal{P}' \leqq \mathcal{P}$, and let $E'' \in \mathcal{P}''$. Then there is a set $E' \in \mathcal{P}'$ such that $E'' \subseteq E'$, and there is a set $E \in \mathcal{P}$ such that $E' \subseteq E$. But $E'' \subseteq E'$ and $E' \subseteq E$ together imply that $E'' \subseteq E$. We've shown that for every $E'' \in \mathcal{P}''$ there is an $E \in \mathcal{P}$ such that $E'' \subseteq E$ — *i.e.*, $\mathcal{P}'' \leqq \mathcal{P}$.

$\leqq$ is antisymmetric, *i.e.*, $[\mathcal{P}' \leqq \mathcal{P} \ \& \ \mathcal{P} \leqq \mathcal{P}'] \Rightarrow \mathcal{P}' = \mathcal{P}$: Assume that $\mathcal{P}' \leqq \mathcal{P}$ and $\mathcal{P} \leqq \mathcal{P}'$. Let $E' \in \mathcal{P}'$; because $\mathcal{P}' \leqq \mathcal{P}$ there is an $E \in \mathcal{P}$ such that $E' \subseteq E$. But then, because $\mathcal{P} \leqq \mathcal{P}'$, there is an $E'' \in \mathcal{P}'$ such that $E \subseteq E''$. We have $E', E'' \in \mathcal{P}'$ and $E' \subseteq E$ and $E \subseteq E''$, and therefore $E' \subseteq E''$. Since both $E' \in \mathcal{P}'$ and $E'' \in \mathcal{P}'$, and $\mathcal{P}'$ is a partition, then either $E' \cap E'' = \emptyset$ or $E' = E''$. We cannot have $E' \cap E'' = \emptyset$, because $E' \subseteq E''$ and $E' \neq \emptyset$. Therefore $E' = E''$. And since $E' \subseteq E \subseteq E''$ and $E' = E''$, we have $E = E'$ — *i.e.*, the *only* $E \in \mathcal{P}$ such that $E' \subseteq E$ is E' itself. Since this is true for each $E' \in \mathcal{P}'$, we have $\mathcal{P}' = \mathcal{P}$, completing the proof.

Recall that in Econ 501B we modeled uncertainty and information in terms of partitions of the state space, and we said that a partition $\mathcal{P}'$ provides more, or better, information than a partition $\mathcal{P}$ if $\mathcal{P}'$ is a refinement of $\mathcal{P}$. If uncertainty (an individual's uncertainty, or the society's uncertainty) about the true state is being resolved over time, and information is not "lost" from one period to the next, then for any $t' > t$ the partition $\mathcal{P}_{t'}$ at time t' would be a refinement of the partition $\mathcal{P}_t$ at time t.

5. Two Manhattan pretzel vendors must decide where to locate their pretzel carts along a given block of Fifth Avenue. Represent the "block of Fifth Avenue" by the unit interval $I = [0,1] \subseteq \mathbb{R}$ — *i.e.*, each vendor chooses a location $x_i \in [0,1]$. The profit π_i of each vendor i depends continuously on *both* vendors' locations — *i.e.*, the function $\pi_i : I \times I \to \mathbb{R}$ is continuous for $i = 1, 2$. Furthermore, each π_i is strictly concave in x_i.

Define an equilibrium in this situation to be a joint action $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in I^2$ that satisfies both

$$\forall x_1 \in I : \pi_1(\widehat{x}) \geqq \pi_1(x_1, \widehat{x}_2) \qquad \text{and} \qquad \forall x_2 \in I : \pi_2(\widehat{x}) \geqq \pi_2(\widehat{x}_1, x_2).$$

In other words, an equilibrium consists of a location for each vendor, with the property that each one's location is best for him given the other's location.

Prove that an equilibrium exists.

Solution:

For each $i = 1, 2$, apply the Maximum Theorem to establish that the vendor's reaction function is continuous. For $i = 1$, let $\varphi : I \twoheadrightarrow I$ be the feasible-set correspondence $\varphi(x_2) = I$, which is constant and therefore continuous, and also nonempty- and compact-valued. Let $\mu_1 : I \to I$ be the reaction function, $\mu_1(x_2) = \text{argmax}_{\varphi(x_2)} \pi_1(x_1, x_2)$ (because π_1 is strictly concave in x_1, μ_1 is singleton-valued, so is indeed a function). Since π_1 is continuous, all the conditions of the Maximum Theorem are satisfied, therefore μ_1 is a closed correspondence, which, since μ_1 is a function, means the function μ_1 is continuous. The same argument also establishes that μ_2 is a continuous function.

Now define a "transition function" $f : I \times I \to I \times I$ as follows:

$$\forall (x_1, x_2) \in I^2 : f(x_1, x_2) = \big(\mu_1(x_2), \mu_2(x_1)\big).$$

The function f is continuous and the set I^2 is nonempty, compact, and convex. The Brouwer Fixed Point Theorem therefore ensures that f has a fixed point, $\widehat{\mathbf{x}} = (\widehat{x}_1 . \widehat{x}_2)$ — a point at which $\mu_1(\widehat{x}_2) = \widehat{x}_1$ and $\mu_2(\widehat{x}_1) = \widehat{x}_2$, *i.e.*, a (Nash) equilibrium.