Economics 519 Exam
University of Arizona
Fall 2014

Closed-book part: Don't consult books or notes until you've handed in your solutions to Problems #1 - #3. After you do consult books or notes, your solutions to Problems #1 - #3 will not be accepted.

1. Prove that if X is a convex set, then the closure of X, denoted by $\overline{X}$, is convex.

2. Our definition of the interior of a set S, denoted S°, is that S° is the union of all open subsets of S (*i.e.*, S° is the largest open subset of S). Prove that a point x is in S° if and only if there is an open ball $B(x, r)$ about x that is contained in S.

3. Our definition of a continuous function is that $f : X \to Y$ is continuous if for any sequence $\{x_n\}$ that converges to a point $\overline{x}$, the sequence $\{f(x_n)\}$ converges to $f(\overline{x})$. From that definition, prove that a function is continuous if and only if the inverse image $f^{-1}(V)$ of any open set $V \subseteq Y$ is an open set in X.

Open-book part: Be sure to turn in your solutions to Problems #1 - #3 before starting this.

4. Provide an example of a metric space (S, d) and a function $f : S \to S$ that is a contraction but does not have a fixed point.

5. Let A be an $n \times n$ matrix. Define a binary relation on $\mathbb{R}^n$ as follows: $x \sim x'$ if and only if $Ax = Ax'$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Assume $rank A = n$. Describe clearly the equivalence classes that constitute the quotient space $\mathbb{R}^n / \sim$, and explain why these are the correct equivalence classes.

(c) Let $n = 3$ and assume $rank A = 1$. Describe clearly the equivalence classes that constitute the quotient space $\mathbb{R}^n / \sim$, and explain why these are the correct equivalence classes.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a concave real function. Let $\{x_n\}$ and $\{y_n\}$ be sequences that satisfy

$$\forall n : y_n = f(x_n) \quad \text{and} \quad \overline{x} = \lim x_n \quad \text{and} \quad \overline{y} = \lim y_n.$$

Prove that $f(\overline{x}) \leqq \overline{y}$. (Note that to prove f is continuous, you would have to prove that $f(\overline{x}) = \overline{y}$. Proving that $f(\overline{x}) \geqq \overline{y}$ is more difficult than $f(\overline{x}) \leqq \overline{y}$; don't try to do that here. Also, ***don't*** try to emulate de la Fuente's proof; that will only mislead you. You'll be better off to not even look at his proof.)