Economics 519 Final Exam
University of Arizona
Fall 2016

Closed-book part: Don't consult books or notes until you've handed in your solutions to Problems #1 - #4. After you do consult books or notes, your solutions to Problems #1 - #4 will not be accepted.

1. Let (X, d) be a metric space (which may or may not be complete) and let $f : X \to X$ be a contraction mapping on X. Prove that f has at most one fixed point — *i.e.*, any fixed point must be unique.

2. Prove that if a sequence $\{x_n\}$ converges to $\overline{x}$ in a metric space (X, d), then every subsequence of $\{x_n\}$ converges to $\overline{x}$.

3. Define the relation $\succsim$ on $\mathbb{R}^2$ by $\mathbf{x}' \succsim \mathbf{x} \;\Leftrightarrow\; [x_1' \geqq x_1 \;\&\; x_2' \geqq x_2]$, and define the strict relation $\succ$ and the equivalence relation $\sim$ from $\succsim$ in the usual way.

(a) In a diagram, depict the weak upper- and lower-contour sets of a typical point $\overline{x} \in \mathbb{R}^2$.

(b) Prove that $\succsim$ is a preorder, *i.e.*, is reflexive and transitive.

(c) Is $\succsim$ a complete preorder? What is the equivalence class $[\overline{x}]$ of a typical $\overline{x} \in \mathbb{R}^2$? Verify your answers.

(d) Let $c \in \mathbb{R}$ be an arbitrary real number, and let $X = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 \leqq c\}$. Is a maximal element for $\succsim$ on X also a maximum element? If yes, provide a proof; if not, provide an example of a maximal element that is not a maximum.

4. Prove that the interior of a convex set in $\mathbb{R}^n$ is convex.

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5. Assume that the sum of any two convex sets in $\mathbb{R}^n$ is convex, and prove by induction that the sum of any finite collection of convex sets in $\mathbb{R}^n$ is convex.

Problem #6 is on the following page

6. Let S be the unit simplex in $\mathbb{R}^2_+$: $S = \{\mathbf{p} \in \mathbb{R}^2_+ \,|\, p_1 + p_2 = 1\}$. For any fixed $\mathring{\mathbf{x}} \in \mathbb{R}^2_+$, define the budget set correspondence $f : S \twoheadrightarrow \mathbb{R}^2_+$ in the usual way:

$$f(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2_+ \,|\, \mathbf{p} \cdot \mathbf{x} \leqq \mathbf{p} \cdot \mathring{\mathbf{x}}\}.$$

(a) Prove that f is a closed correspondence, *i.e.*, that f has a closed graph. For this you may use, without proving them, such properties of real sequences as these:

If $\lim x_n = \overline{x}$ and $\lim y_n = \overline{y}$, then $\lim(x_n + y_n) = \overline{x} + \overline{y}$ and $\lim x_n y_n = \overline{x}\,\overline{y}$.
If $\lim x_n = \overline{x}$ and $\forall n \in \mathbb{N} : x_n \leqq a$, then $\overline{x} \leqq a$.

(b) Now assume that $\beta > \max\{\mathring{x}_1, \mathring{x}_2\}$ and that $K = \{\mathbf{x} \in \mathbb{R}^2_+ \,|\, 0 \leqq x_1 \leqq \beta,\ 0 \leqq x_2 \leqq \beta\}$. Define $g : S \twoheadrightarrow K$ by $g(\mathbf{p}) = f(\mathbf{p}) \cap K$. Using any results from the lecture notes that you need, prove that g is upper-hemicontinuous.