Economics 519 Final Exam
University of Arizona
Fall 2017

Don't consult books or notes during the exam. The last page of the exam contains some definitions that you might find useful.

1. Consult the definitions on the last page of the exam.

(a) Prove that if $||\cdot||$ is a norm on a vector space V, then the function $d : V \times V \to \mathbb{R}$ defined by $d(x, x') := ||x - x'||$ is a metric on V.

(b) Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the discrete metric on $\mathbb{R}^n$. If there exists a norm $||\cdot||$ on $\mathbb{R}^n$ for which $d(x, x') = ||x - x'||$, identify any such norm. If there is no such norm, verify that.

2. A real number x is rational if it can be expressed as the ratio of two integers — *i.e.*, if $x = k/m$ for some $k, m \in \mathbb{Z}$. Prove by induction that for any $n \in \mathbb{N}$, the sum of n rational numbers is rational. (You may use the fact that $\mathbb{Z}$, the set of integers, is closed under the operations of addition and multiplication.)

3. Assume that X is a convex subset of a vector space V and that $f : X \to \mathbb{R}$ is a concave function. Let S be the set of solutions of the constrained maximization problem

$$\max f(x) \,\text{subject to}\, x \in X.$$

Must the set S be convex? If so, prove it; if not, provide a counterexample.

4. A relation R on a set $\mathcal{Z}$ is a **partial order** or a **partial ordering** of $\mathcal{Z}$ if R is reflexive, transitive, and antisymmetric. Two examples of partial orders that we've encountered are the vector partial order on $\mathbb{R}^n$ and the subset partial order $\subseteq$ on the set 2^X of all subsets of a set X. (In the second case, 2^X is playing the role of $\mathcal{Z}$.) Now let X be a set and let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of X; $\mathcal{P}'$ is said to be a **refinement** of $\mathcal{P}$, denoted $\mathcal{P}' \leqq \mathcal{P}$, if for every $E' \in \mathcal{P}'$ there is an $E \in \mathcal{P}$ such that $E' \subseteq E$. We also say that $\mathcal{P}'$ is **at least as informative** as $\mathcal{P}$. Prove that the refinement relation $\leqq$ is a partial ordering of the set of all partitions of X. (It could be helpful to note that if $\sim$ and $\sim'$ are the equivalence relations defined by $\mathcal{P}$ and $\mathcal{P}'$, then $\mathcal{P}' \leqq \mathcal{P}$ if and only if $x \sim' y \Rightarrow x \sim y$ for all $x, y \in X$.)

5. Two Manhattan pretzel vendors must decide where to locate their pretzel carts along a given block of Fifth Avenue. Represent the "block of Fifth Avenue" by the unit interval $I = [0,1] \subseteq \mathbb{R}$ — *i.e.*, each vendor chooses a location $x_i \in [0,1]$. The profit π_i of each vendor i depends continuously on *both* vendors' locations — *i.e.*, the function $\pi_i : I \times I \to \mathbb{R}$ is continuous for $i = 1, 2$. Furthermore, each π_i is strictly concave in x_i.

Define an equilibrium in this situation to be a joint action $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in I^2$ that satisfies both

$$\forall x_1 \in I : \pi_1(\widehat{x}) \geqq \pi_1(x_1, \widehat{x}_2) \qquad \text{and} \qquad \forall x_2 \in I : \pi_2(\widehat{x}) \geqq \pi_2(\widehat{x}_1, x_2).$$

In other words, an equilibrium consists of a location for each vendor, with the property that each one's location is best for him given the other's location.

Prove that an equilibrium exists.

Definitions

Definition: Let V be a vector space. A function $||\cdot|| : V \to \mathbb{R}$ is a **norm** on V if it satisfies the following four conditions:

(N1) $\forall \mathbf{x} \in V : ||\mathbf{x}|| \geqq 0$;

(N2) $\forall \mathbf{x} \in V : ||\mathbf{x}|| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;

(N3) $\forall \mathbf{x}, \mathbf{y} \in V : ||\mathbf{x} + \mathbf{y}|| \leqq ||\mathbf{x}|| + ||\mathbf{y}||$;

(N4) $\forall \alpha \in \mathbb{R}, \mathbf{x} \in V : ||\alpha \mathbf{x}|| = |\alpha| \, ||\mathbf{x}||$.

Definition: Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbb{R}$ that satisfies following four conditions:

(D1) $d(x, x') \geqq 0$;

(D2) $d(x, x') = 0 \Leftrightarrow x = x'$;

(D3) $d(x, x') = d(x', x)$;

(D4) $d(x, x'') \leqq d(x, x') + d(x', x'')$.

The pair (X, d) is called a **metric space**.

Definition: A relation R on a set $\mathcal{Z}$ is **antisymmetric** if $[xRy \,\&\, yRx] \Rightarrow x = y$.